

Breaking of Global Gauge Symmetries on a Spatial-temporal Subdomain

Zaixing Huang

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Abstract It is generally considered that global gauge symmetry of an action integral over a finite spatial-temporal domain as a whole holds in any subdomain of this domain. However, in this paper we demonstrate that when the so-called localized assumption is no longer valid, the global gauge symmetry will break in the subdomain of this spatial-temporal domain, due to occurrence of a new source called the nonlocal residual. The nonlocal residual is correlated with a gauge field and satisfies the zero mean condition with respect to whole spatial-temporal domain. An example is given to support this conclusion.

Keywords Lagrange field theory · Global gauge symmetry · Localized assumption · Nonlocal residual · Gauge field

1 Introduction

Should it necessarily be the case that a statement holds for every part of a field or body if it holds for the whole field or body? The answer is in general negative. The examples in physics provided by Edelen [1] show that an integral statement for the whole field or body (for shorthand, we will use the “body” and “field” without distinction in the latter) is true, and yet when the exactly same statement is made for a subset of the body it ceases to be valid. In fact, there are some physical phenomena in which it is not always advisable to write a mathematical representation for a part of a body that has the same form and uses the same functions as occur in the corresponding formula for the whole body. As a result, the assertion that a statement on a body as a whole is valid to each part of the body is merely an assumption in physics. This is the so-called localized assumption [1–3], which manifests itself in the following procedures:

Z. Huang (✉)
College of Aerospace Engineering, Nanjing University of Aeronautics and Astronautics, Nanjing
210016, People’s Republic of China
e-mail: huangzx@nuaa.edu.cn

(i) Statement of a global equilibrium for a state of the body or field, that is

$$\int_{\Omega} \underline{T}(\underline{x}) dV(\underline{x}) = 0, \quad (1)$$

where \underline{x} denotes the space-time coordinate. $\underline{T}(\underline{x})$ is a state function, which can be either scalar, vector or tensor depending on circumstance. Ω is a bounded space-time domain occupied by the body or field.

(ii) Assume that (1) is also valid for every part V of Ω with the same function $\underline{T}(\underline{x})$, that is

$$\int_V \underline{T}(\underline{x}) dV(\underline{x}) = 0. \quad (2)$$

(iii) In terms of the localized theorem [4], the local equation can be, from (2), given as follows:

$$\underline{T}(\underline{x}) = 0. \quad (3)$$

The step from (i) to (ii) is just an embodiment of the localized assumption. This assumption has been adopted all along in physics. By means of it, an integral representation can be conveniently transformed into a differential equation. If the localized assumption is abandoned, then a way will lead to the so-called nonlocal theories in which the relevant physical formulations are generally given by a group of integro-differential equations.

At present, there are two ways to comprehend symmetries of the Lagrange field. One is based on invariance under the transformations of an action integral; the other is based on the invariance under the transformations of the Lagrangian itself [5]. For the gauge symmetry, only the gauge transformation of the Lagrangian is concerned in literatures [6, 7]. Maybe it is due to the fact that under the gauge transformation, the invariance of the Lagrangian is regarded to be equivalent to the invariance of the action integral. However, if an elaborate analysis is made, one will find that the equivalence between the invariance of the Lagrangian and that of the action integral is guaranteed by the localized assumption mentioned above. According to this assumption, the invariance of the action integral defined on a domain as a whole also inevitably holds for any part of this domain no matter how small it is. Therefore, some problems to be worth asking are what is the reason for needing such an assumption, and what will happen when the localization hypothesis is excluded. To answer these questions is the subject of this paper.

The premise of this paper contains three main propositions: (1) A body or physical field is supposed to distribute over a bounded space-time domain; (2) Under the gauge transformations, the symmetry of a Lagrange field should be comprehended as the invariance of the action integral of the Lagrangian, not the Lagrangian itself; (3) The localized assumption is considered to be no avail. On the basis of these premises, emphasis of this paper is focused on how to express, after the localized assumption fails, the “source” causing the global gauge symmetry broken in some local subdomain of a bounded space-time domain on which the action integral is defined.

The paper is divided into five parts. The first section is an introduction, which gives the background of this paper. In the second section, we discuss the conservation flux characterizing the global gauge symmetry of a Lagrangian field, under the condition that the localized assumption fails. A new source called the “nonlocal residual” is introduced, which causes the conservation flux not to be a constant. In the third section, through investigating the local gauge symmetry of an action integral, a gauge field is introduced, and then it is extended to the case of the global gauge transformation. On the basis of this, the correlation between the

nonlocal residual and the gauge field is determined. In the fourth section, a complex scalar field as an example is used to show the limitation of the localized assumption. By means of the nonlocal balance equation of the complex scalar field, an explicitly relation between the nonlocal residual and the gauge field is given. Finally, some discussions on the results obtained in this paper are drawn.

2 Global Gauge Symmetry of the Action Integral after the Localized Assumption Fails

Suppose that x^μ ($\mu = 1, 2, \dots, k$) and φ_α ($\alpha = 1, 2, \dots, n$) are the space-time coordinates and the variables of field, respectively. The action integral $A[\varphi_\alpha]$, defined on a bounded space-time domain $\Omega \subset E^n$, takes the form as follows

$$A[\varphi_\alpha] = \int_{\Omega} L(x^\mu, \varphi_\alpha, \varphi_{\alpha,v}) dV(x^\mu), \tag{4}$$

where $L(x^\mu, \varphi_\alpha, \varphi_{\alpha,\sigma})$ is the Lagrangian density function, or simply called the Lagrangian. Consider an infinitesimal gauge transformation

$$\varphi_\alpha(x^\mu) \rightarrow \tilde{\varphi}_\alpha(x^\mu) = \varphi_\alpha(x^\mu) + \delta\varphi_\alpha(x^\mu). \tag{5}$$

The action integral $A[\varphi_\alpha]$ is said to be gauge symmetry if it is form-invariant with respect to the infinitesimal gauge transformation, i.e.,¹

$$\int_{\Omega} L(x^\mu, \tilde{\varphi}_\alpha, \tilde{\varphi}_{\alpha,v}) dV(x^\mu) = \int_{\Omega} L(x^\mu, \varphi_\alpha, \varphi_{\alpha,v}) dV(x^\mu). \tag{6}$$

After rearrangement, (6) becomes

$$\int_{\Omega} [L(x^\mu, \tilde{\varphi}_\alpha, \tilde{\varphi}_{\alpha,v}) - L(x^\mu, \varphi_\alpha, \varphi_{\alpha,v})] dV(x^\mu) = 0. \tag{7}$$

It is easy to calculate that

$$\begin{aligned} \delta L &= L(x^\mu, \tilde{\varphi}_\alpha, \tilde{\varphi}_{\alpha,v}) - L(x^\mu, \varphi_\alpha, \varphi_{\alpha,v}) \\ &= \frac{\partial L}{\partial \varphi_\alpha} \delta\varphi_\alpha + \frac{\partial L}{\partial \varphi_{\alpha,v}} \delta\varphi_{\alpha,v} \\ &= \frac{\partial L}{\partial \varphi_\alpha} \delta\varphi_\alpha + \left(\frac{\partial L}{\partial \varphi_{\alpha,v}} \delta\varphi_\alpha \right)_{,v} - \left(\frac{\partial L}{\partial \varphi_{\alpha,v}} \right)_{,v} \delta\varphi_\alpha \\ &= \left(\frac{\partial L}{\partial \varphi_{\alpha,v}} \delta\varphi_\alpha \right)_{,v} + \left[\frac{\partial L}{\partial \varphi_\alpha} - \left(\frac{\partial L}{\partial \varphi_{\alpha,v}} \right)_{,v} \right] \delta\varphi_\alpha. \end{aligned} \tag{8}$$

¹ Addition a four-divergence to the Lagrangian does not cause the Euler-Lagrange equation (motion equation) changed. Therefore, (6) can be replaced by a more general equality:

$$\int_{\Omega} L(x^\mu, \tilde{\varphi}_\alpha, \tilde{\varphi}_{\alpha,v}) dV(x^\mu) + \int_{\Omega} \Psi_{,\gamma}^\gamma(x^\mu, \tilde{\varphi}_\alpha, \tilde{\varphi}_{\alpha,v}) dV(x^\mu) = \int_{\Omega} L(x^\mu, \varphi_\alpha, \varphi_{\alpha,v}) dV(x^\mu).$$

However, to the discussion in this paper, omitting this four-divergence has no influence on the latter results.

Here, repeated indices mean summation. Substituting (8) into (7) yields

$$\int_{\Omega} \left\{ \left(\frac{\partial L}{\partial \varphi_{\alpha, \nu}} \delta \varphi_{\alpha} \right)_{, \nu} + \left[\frac{\partial L}{\partial \varphi_{\alpha}} - \left(\frac{\partial L}{\partial \varphi_{\alpha, \nu}} \right)_{, \nu} \right] \delta \varphi_{\alpha} \right\} dV(x^{\mu}) = 0. \tag{9}$$

Assume that the action integral $A[\varphi_{\alpha}]$ takes an extremum on φ_{α} . Then, the Lagrangian necessarily satisfies the Euler-Lagrange equation (motion equation) below:

$$\frac{\partial L}{\partial \varphi_{\alpha}} - \left(\frac{\partial L}{\partial \varphi_{\alpha, \nu}} \right)_{, \nu} = 0. \tag{10}$$

Inserting (10) in (9) leads to

$$\int_{\Omega} \left(\frac{\partial L}{\partial \varphi_{\alpha, \nu}} \delta \varphi_{\alpha} \right)_{, \nu} dV(x^{\mu}) = 0. \tag{11}$$

If (5) belongs to a finite Lie group of infinitesimal transformations, according to the representation of Lie group, $\delta \varphi_{\alpha}$ can be written as [8–10]

$$\delta \varphi_{\alpha} = \varepsilon^{\beta} \Phi_{\beta \alpha}, \tag{12}$$

where ε^{β} is an infinitesimal parameter independent of the space-time coordinates and $\Phi_{\beta \alpha}$ is the infinitesimal generator of Lie group of transformations. Substituting (12) into (11) yields

$$\varepsilon^{\beta} \int_{\Omega} \left(\frac{\partial L}{\partial \varphi_{\alpha, \nu}} \Phi_{\beta \alpha} \right)_{, \nu} dV(x^{\mu}) = 0. \tag{13}$$

Due to ε^{β} taking an arbitrary value, (13) reduces to

$$\int_{\Omega} \left(\frac{\partial L}{\partial \varphi_{\alpha, \nu}} \Phi_{\beta \alpha} \right)_{, \nu} dV(x^{\mu}) = 0. \tag{14}$$

If the localized assumption is true, then (14) is also valid for any $V \subset \Omega$. That is

$$\int_V \left(\frac{\partial L}{\partial \varphi_{\alpha, \nu}} \Phi_{\beta \alpha} \right)_{, \nu} dV(x^{\mu}) = 0. \tag{15}$$

Thus, applying the localization theorem [4] to (15) gives

$$\left(\frac{\partial L}{\partial \varphi_{\alpha, \nu}} \Phi_{\beta \alpha} \right)_{, \nu} = 0, \tag{16}$$

which is a consequence of the Noether’s theorem [8, 9], under the infinitesimal gauge transformation. However, we have not yet a sufficient reason to need such a prior condition as the localized assumption. Therefore, if the localization assumption is no longer considered to be valid, then we can not directly derive (15) from (14), instead, a new term will occur in (15) such that

$$\int_V \left(\frac{\partial L}{\partial \varphi_{\alpha, \nu}} \Phi_{\beta \alpha} \right)_{, \nu} dV(x^{\mu}) = R_{\beta}(V). \tag{17}$$

Obviously, $R_\beta(V)$ is a generalized measure function defined on V . Assume it is absolutely continuous with respect to V . So according to the Radon-Nikodym theorem [11], $R_\beta(V)$ can be represented as

$$R_\beta(V) = \int_V F_\beta(x^\mu) dV(x^\mu), \quad (18)$$

where $F_\beta = F_\beta(x^\mu)$ is called the nonlocal residual or localization residual. Substituting (18) into (17) yields

$$\int_V \left(\frac{\partial L}{\partial \varphi_{\alpha,v}} \Phi_{\beta\alpha} \right)_{,v} dV(x^\mu) = \int_V F_\beta(x^\mu) dV(x^\mu). \quad (19)$$

Since V can be arbitrarily chosen, so (19) holds if and only if

$$\left(\frac{\partial L}{\partial \varphi_{\alpha,v}} \Phi_{\beta\alpha} \right)_{,v} = F_\beta. \quad (20)$$

Equation (20) is referred to as the nonlocal balance equation, which is a generalization of the Noether's formulation under the global gauge transformation. Let

$$J_\beta^v = \frac{\partial L}{\partial \varphi_{\alpha,v}} \Phi_{\beta\alpha}, \quad (21)$$

where J_β^v is called the conservation flux. Accordingly, (20) can be also shortly written as

$$J_{\beta,v}^v = F_\beta. \quad (22)$$

When $V = \Omega$, comparison of (19) with (14) leads to the so-called "zero mean condition",

$$\int_\Omega F_\beta(x^\mu) dV(x^\mu) = 0. \quad (23)$$

In general, F_β does not vanish everywhere. So (23) forms a constraint to (20) or (22). These equations show that, although global effects of F_β on Ω as a whole are null, F_β has in essence influences on the local conservation flux. It makes the local conservation flux not being a constant. Therefore, under the global gauge transformation, if the action integral over a space-time domain as a whole has symmetry, this symmetry will lose in any subdomain of this domain when the localized assumption fails. This means that the global gauge symmetry of the action integral locally breaks. In this symmetry breaking, the nonlocal residual acts as a source.

3 Correlation Between the Nonlocal Residual and Gauge Field

In physics, the nonlocal residual is considered to originate from self-interactions among different local regions within a field or body [1, 2]. These self-interactions induce, in the sub-domain of the field, the symmetry breaking of the action integral. Thereby, a problem arises from how to characterize the nonlocal residual in physics. As well known, under a local gauge transformation, the gauge symmetry of an action integral will lose unless a gauge field is introduced by means of the Yang-Mills minimal coupling principle [8, 9]. The new gauge field acts also as a source of the conservation flux in the so-called local gauge

invariance. Such facts hint us that there are probably some correlations between the nonlocal residual and the gauge field.

The infinitesimal gauge transformation can also be represented as [8]:

$$\varphi_\alpha(x^\mu) \rightarrow \tilde{\varphi}_\alpha(x^\mu) = (1 + \varepsilon^\beta \Phi_\beta)\varphi_\alpha(x^\mu), \tag{24}$$

where ε^β is a linear operator, which is written as

$$\Phi_\beta = \left. \frac{\partial}{\partial \varepsilon^\beta} \right|_{\varepsilon^\beta=0}. \tag{25}$$

If ε^β is independent of the space-time coordinates, the transformation (24) is called the global gauge transformation. Or else, it refers to the local gauge transformation. Under the global gauge transformation, the derivative of the field variable φ_α with respect to the coordinate x^ν has the same form as (24), i.e.,

$$\varphi_{\alpha,\nu}(x^\mu) \rightarrow \tilde{\varphi}_{\alpha,\nu}(x^\mu) = (1 + \varepsilon^\beta \Phi_\beta)\varphi_{\alpha,\nu}(x^\mu). \tag{26}$$

If the action integral is unchanged with the global gauge transformation,² then (22) will be given once again, and the conservation flux can be written as

$$J_\beta^\nu = \frac{\partial L}{\partial \varphi_{\alpha,\nu}} \Phi_\beta \varphi_\alpha. \tag{27}$$

Of course, if the localized assumption is supposed to be valid, (22) will then reduce to

$$J_{\beta,\nu}^\nu = 0. \tag{28}$$

However, under the local gauge transformation, neither (22) nor (28) holds because the action integral is no longer invariant, as

$$\varphi_{\alpha,\nu}(x^\mu) \rightarrow \tilde{\varphi}_{\alpha,\nu}(x^\mu) = (1 + \varepsilon^\beta \Phi_\beta)\varphi_{\alpha,\nu}(x^\mu) + (\varepsilon^\beta \Phi_\beta)_{,\nu}\varphi_\alpha(x^\mu), \tag{29}$$

being different from the transformation given by (24) in form. To construct a local gauge invariant theory, a new field, called the gauge field, should be introduced to render the action integral invariant. According to the Yang-Mills minimal coupling principle [9], we define a covariant derivative as follows:

$$\frac{D}{Dx^\nu} = \frac{\partial}{\partial x^\nu} + eA_\nu, \tag{30}$$

where e denotes the coupling constant. A_ν refers to the gauge field, which transforms according to

$$A_\nu(x^\mu) \rightarrow \tilde{A}_\nu(x^\mu) = A_\nu(x^\mu) + \varepsilon^\beta [\Phi_\beta, A_\nu] - \frac{1}{e} \frac{\partial(\varepsilon^\beta \Phi_\beta)}{\partial x^\nu}, \tag{31}$$

in which $[\Phi_\beta, A_\nu]$ is defined as

$$[\Phi_\beta, A_\nu] = \Phi_\beta A_\nu - A_\nu \Phi_\beta. \tag{32}$$

²It should be noted that when the localized assumption fails, the invariance of action integral is not equivalent to the invariance of Lagrangian under a global gauge transformation.

Under the local gauge transformation, it is easy to verify that

$$\frac{D\varphi_\alpha}{Dx^\nu} \rightarrow \frac{\tilde{D}\tilde{\varphi}_\alpha}{\tilde{D}x^\nu} = (1 + \varepsilon^\beta \Phi_\beta) \frac{D\varphi_\alpha}{Dx^\nu}, \tag{33}$$

which has the same form as (26).

In terms of the Yang-Mills minimal replacing principle [9], we use the covariant derivative D/Dx^ν instead of the common derivative $\partial/\partial x^\nu$ in the Lagrangian. As thus, under the local gauge transformation, the action integral will remain unchanged. That is,

$$\int_\Omega L\left(x^\mu, \tilde{\varphi}_\alpha, \frac{\tilde{D}\tilde{\varphi}_\alpha}{\tilde{D}x^\nu}\right) dV(x^\mu) = \int_\Omega L\left(x^\mu, \varphi_\alpha, \frac{D\varphi_\alpha}{Dx^\nu}\right) dV(x^\mu). \tag{34}$$

In order to derive the conservation flux from (34), we need to calculate

$$\begin{aligned} \delta L &= L\left(x^\mu, \tilde{\varphi}_\alpha, \frac{\tilde{D}\tilde{\varphi}_\alpha}{\tilde{D}x^\nu}\right) - L\left(x^\mu, \varphi_\alpha, \frac{D\varphi_\alpha}{Dx^\nu}\right) \\ &= \frac{\partial L}{\partial \varphi_\alpha} \delta \varphi_\alpha + \frac{\partial L}{\partial \left(\frac{D\varphi_\alpha}{Dx^\nu}\right)} \delta \left(\frac{D\varphi_\alpha}{Dx^\nu}\right) \\ &= \frac{\partial L}{\partial \varphi_\alpha} (\varepsilon^\beta \Phi_\beta \varphi_\alpha) + \frac{\partial L}{\partial \varphi_{\alpha,\nu}} \left(\varepsilon^\beta \Phi_\beta \frac{D\varphi_\alpha}{Dx^\nu}\right) \\ &= \frac{\partial L}{\partial \varphi_\alpha} \varepsilon^\beta \Phi_\beta \varphi_\alpha + \frac{\partial L}{\partial \varphi_{\alpha,\nu}} \varepsilon^\beta \Phi_\beta (\varphi_{\alpha,\nu} + e A_\nu \varphi_\alpha) \\ &= \varepsilon^\beta \frac{\partial L}{\partial \varphi_\alpha} \Phi_\beta \varphi_\alpha + \varepsilon^\beta \frac{\partial L}{\partial \varphi_{\alpha,\nu}} \Phi_\beta \varphi_{\alpha,\nu} + e \varepsilon^\beta \frac{\partial L}{\partial \varphi_{\alpha,\nu}} \Phi_\beta A_\nu \varphi_\alpha \\ &= \varepsilon^\beta \left(\frac{\partial L}{\partial \varphi_{\alpha,\nu}}\right)_{,\nu} \Phi_\beta \varphi_\alpha + \varepsilon^\beta \frac{\partial L}{\partial \varphi_{\alpha,\nu}} (\Phi_\beta \varphi_\alpha)_{,\nu} + e \varepsilon^\beta \frac{\partial L}{\partial \varphi_{\alpha,\nu}} \Phi_\beta A_\nu \varphi_\alpha \\ &= \varepsilon^\beta \left(\frac{\partial L}{\partial \varphi_{\alpha,\nu}} \Phi_\beta \varphi_\alpha\right)_{,\nu} + e \varepsilon^\beta \frac{\partial L}{\partial \varphi_{\alpha,\nu}} \Phi_\beta A_\nu \varphi_\alpha \\ &= \varepsilon^\beta J_{\beta,\nu}^v + e \varepsilon^\beta \frac{\partial L}{\partial \varphi_{\alpha,\nu}} \Phi_\beta A_\nu \varphi_\alpha. \end{aligned} \tag{35}$$

The last equals sign is due to (27). Inserting (35) in (34) leads to

$$\int_\Omega \varepsilon^\beta \left(J_{\beta,\nu}^v + e \frac{\partial L}{\partial \varphi_{\alpha,\nu}} \Phi_\beta A_\nu \varphi_\alpha\right) dV(x^\mu) = 0. \tag{36}$$

Because ε^β in (36) can be arbitrarily choose, we have

$$J_{\beta,\nu}^v + e \frac{\partial L}{\partial \varphi_{\alpha,\nu}} \Phi_\beta A_\nu \varphi_\alpha = 0. \tag{37}$$

It is interesting to notice the distinguish of ε^β in (13) and in (36). In (13), ε^β is independent of the space-time coordinates. So it can be moved into the exterior of integral symbol. This makes us to derive (16) from (14) only by way of the localization theorem [4]. On the contrary, ε^β in (36) can not be moved into the exterior of integral symbol due to it depending

on coordinates. However, we can directly obtain (37) from (36) in terms of the variational lemma [10], not needing to rely on the localization theorem.

In fact, (37) also holds for the global gauge transformation. Under this circumstance, ε^β is a constant, and (31) reduces to

$$A_\nu(x^\mu) \rightarrow \tilde{A}_\nu(x^\mu) = A_\nu(x^\mu) + \varepsilon^\beta [\Phi_\beta, A_\nu]. \tag{38}$$

When ε^β is an infinitesimal constant, the second term of (37) also satisfies the zero mean condition. In order to verify this argument, we firstly need to prove that the integral of $J_{\beta,\nu}^v$ in (37) on Ω is equal to zero. For this, let us write out the necessary condition of the action integral taking the extrema,

$$\int_{\Omega} \left[\frac{\partial L}{\partial \varphi_\alpha} - \left(\frac{\partial L}{\partial \varphi_{\alpha,\nu}} \right)_{,\nu} \right] \delta \varphi_\alpha dV(x^\mu) + \int_{\partial \Omega} \frac{\partial L}{\partial \varphi_{\alpha,\nu}} \delta \varphi_\alpha n_\nu dS(x^\mu) = 0. \tag{39}$$

On the boundary of Ω , regardless of whether $\delta \varphi_\alpha$ is zero or not zero, (10) is always valid. Accordingly, (39) is simplified to

$$\int_{\partial \Omega} \frac{\partial L}{\partial \varphi_{\alpha,\nu}} \delta \varphi_\alpha n_\nu dS(x^\mu) = 0. \tag{40}$$

Applying the divergence theorem to (40) leads to

$$\int_{\Omega} \left(\frac{\partial L}{\partial \varphi_{\alpha,\nu}} \delta \varphi_\alpha \right)_{,\nu} dV(x^\mu) = 0. \tag{41}$$

Because $\delta \varphi_\alpha$ is arbitrary, a selection is let $\delta \varphi_\alpha = \varepsilon^\beta \Phi_\beta \varphi_\alpha$ and let ε^β be an infinitesimal constant. As a result, (41) reduces to

$$\int_{\Omega} \left(\frac{\partial L}{\partial \varphi_{\alpha,\nu}} \Phi_\beta \varphi_\alpha \right)_{,\nu} dV(x^\mu) = 0. \tag{42}$$

By virtue of (27), (42) can be also written as

$$\int_{\Omega} J_{\beta,\nu}^v dV(x^\mu) = 0. \tag{43}$$

Taking the integral for (37) on Ω and using (43), we have

$$\int_{\Omega} e \frac{\partial L}{\partial \varphi_{\alpha,\nu}} \Phi_\beta A_\nu \varphi_\alpha dV(x^\mu) = 0. \tag{44}$$

This shows that the integrand in (44) also satisfies the zero mean condition. By way of this conclusion, comparing (37) with (22) will lead to

$$F_\beta = e \frac{\partial L}{\partial \varphi_{\alpha,\nu}} \Phi_\beta A_\nu \varphi_\alpha. \tag{45}$$

Therefore, the nonlocal residual surely has a natural connection with the gauge field. In general, when the localized assumption is available, it is meaningless in physics to introduce the gauge field to describe the global gauge invariance. However, if the localized assumption fails, then the gauge field induced by the global gauge invariance physically becomes feasible. It can be used to characterize the nonlocal residual, just as seen from (45).

4 An Example: A Gauge Induced by the Global Gauge Invariance of Action Integral

For convenience, in this section we will use the following notations:

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \quad \partial^\mu = g^{\mu\nu} \partial_\nu, \tag{46}$$

$$\square = \partial_\mu \partial^\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2, \tag{47}$$

where $g^{\mu\nu}$ refers to the metric tensor and \square denotes the d'Alembertian operator.

Consider a complex scalar field. Because the action integral should be real, so the Lagrangian of this complex scalar field is supposed to have the form below:

$$L = \partial_\mu \varphi \partial^\mu \varphi^* - m^2 \varphi \varphi^* - V(\varphi \varphi^*) - U(\partial_\mu \varphi \partial^\mu \varphi^*). \tag{48}$$

Here, φ and φ^* are a pair of conjugate complex variables. Assume $U(\partial_\mu \varphi \partial^\mu \varphi^*)$ can be written as:

$$U(\partial_\mu \varphi \partial^\mu \varphi^*) = \lambda \left[\partial_\mu \varphi \partial^\mu \varphi^* - \frac{1}{V_\Omega} \int_\Omega \partial_\mu \varphi \partial^\mu \varphi^* dV \right], \tag{49}$$

where λ is called the coupling coefficient and V_Ω is the volume of Ω . It is easy to show that

$$\int_\Omega U(\partial_\mu \varphi \partial^\mu \varphi^*) dV = 0. \tag{50}$$

Due to the equality above, $U(\partial_\mu \varphi \partial^\mu \varphi^*)$ may be interpreted as a fluctuation of self-energy of field over the space-time domain Ω . Clearly, both the Lagrangian and its action integral are invariant under the global gauge transformation

$$\varphi \rightarrow e^{-i\phi} \varphi, \quad \varphi^* \rightarrow e^{i\phi} \varphi^*, \tag{51}$$

where ϕ is a real constant. For a long time, there are two ways of comprehending the gauge invariance of a Lagrange field [5]. One is based on the invariance of the Lagrangian; the other is based on the invariance of the action integral of the Lagrangian. When the localized assumption is no longer valid, the gauge invariance of the Lagrangian is included in the gauge invariance of the action integral. So under the gauge transformation, it is of more generality to comprehend the symmetry of a Lagrange field as the invariance of the action integral. Let

$$L_0 = \partial_\mu \varphi \partial^\mu \varphi^* - m^2 \varphi \varphi^* - V(\varphi \varphi^*). \tag{52}$$

Then (48) can be written as

$$L = L_0 - U(\partial_\mu \varphi \partial^\mu \varphi^*). \tag{53}$$

Due to (50), L and L_0 have the same action integral on Ω . Therefore, the Euler-Lagrange equations derived from L and L_0 have the same expression, which read

$$(\square + m^2)\varphi = -\frac{\partial V}{\partial \varphi}, \quad (\square + m^2)\varphi^* = -\frac{\partial V}{\partial \varphi^*}, \tag{54}$$

which are two Klein-Gordon equations. The infinitesimal form of the transformations (51) can be represented as

$$\delta\varphi = -i\phi\varphi, \quad \delta\varphi^* = i\phi\varphi^*, \tag{55}$$

and so

$$\delta(\partial^\mu \varphi) = -i\phi(\partial^\mu \varphi), \quad \delta(\partial^\mu \varphi^*) = i\phi(\partial^\mu \varphi^*). \tag{56}$$

Under these infinitesimal transformations, as the same as deriving (54), we can obtain the equality below:

$$\int_{\Omega} i \left[\varphi^* \frac{\partial L}{\partial(\partial_\mu \varphi^*)} - \varphi \frac{\partial L}{\partial(\partial_\mu \varphi)} \right]_{,\mu} dV = \int_{\Omega} i \left[\varphi^* \frac{\partial L_0}{\partial(\partial_\mu \varphi^*)} - \varphi \frac{\partial L_0}{\partial(\partial_\mu \varphi)} \right]_{,\mu} dV. \tag{57}$$

With (52), the right-hand term of (57) is written as

$$\int_{\Omega} i \left[\varphi^* \frac{\partial L_0}{\partial(\partial_\mu \varphi^*)} - \varphi \frac{\partial L_0}{\partial(\partial_\mu \varphi)} \right]_{,\mu} dV = \int_{\Omega} i(\varphi^* \partial_\mu \partial^\mu \varphi - \varphi \partial_\mu \partial^\mu \varphi^*) dV. \tag{58}$$

It follows immediately from (54) that we have

$$\int_{\Omega} i \left[\varphi^* \frac{\partial L_0}{\partial(\partial_\mu \varphi^*)} - \varphi \frac{\partial L_0}{\partial(\partial_\mu \varphi)} \right]_{,\mu} dV = \int_{\Omega} i \left(\varphi \frac{\partial V}{\partial_\mu \varphi} - \varphi^* \frac{\partial V}{\partial_\mu \varphi^*} \right) dV = 0. \tag{59}$$

Let

$$J^\mu = i \left[\varphi^* \frac{\partial L}{\partial(\partial_\mu \varphi^*)} - \varphi \frac{\partial L}{\partial(\partial_\mu \varphi)} \right], \tag{60}$$

which denotes the conservation flux of L . Thus, (57) becomes

$$\int_{\Omega} J^\mu_{,\mu} dV = \int_{\Omega} i \left[\varphi^* \frac{\partial L_0}{\partial(\partial_\mu \varphi^*)} - \varphi \frac{\partial L_0}{\partial(\partial_\mu \varphi)} \right]_{,\mu} dV. \tag{61}$$

Inserting (59) in (61) yields

$$\int_{\Omega} J^\mu_{,\mu} dV = 0. \tag{62}$$

If the localized assumption holds, from (62) we immediately obtain

$$J^\mu_{,\mu} = i \left[\varphi^* \frac{\partial L}{\partial(\partial_\mu \varphi^*)} - \varphi \frac{\partial L}{\partial(\partial_\mu \varphi)} \right]_{,\mu} = 0. \tag{63}$$

However, substituting (48) into (63), we have

$$\begin{aligned} J^\mu_{,\mu} &= i \left[\varphi^* \frac{\partial L}{\partial(\partial_\mu \varphi^*)} - \varphi \frac{\partial L}{\partial(\partial_\mu \varphi)} \right]_{,\mu} \\ &= \frac{i\lambda}{V_\Omega} \left[\varphi^* \frac{\delta}{\delta(\partial_\mu \varphi^*)} \int_{\Omega} \partial_\nu \varphi^* \partial^\nu \varphi dV - \varphi \frac{\delta}{\delta(\partial_\mu \varphi)} \int_{\Omega} \partial_\nu \varphi \partial^\nu \varphi^* dV \right]_{,\mu} \\ &= \frac{i\lambda}{V_\Omega} \left(\partial_\mu \varphi^* \int_{\Omega} \partial^\mu \varphi dV - \partial_\mu \varphi \int_{\Omega} \partial^\mu \varphi^* dV \right), \end{aligned} \tag{64}$$

where $\delta/\delta(\cdot)$ refers to the Frechet derivative. In general, the right-hand term of (64) is not equal to zero. Therefore, (63) is contradicted with (64). This only shows the localized assumption is on longer valid. Consequently, By introducing the nonlocal residual, (62) can

be transformed to the differential equation below:

$$J_{,\mu}^{\mu} = F. \quad (65)$$

Comparing (64) with (65) gives rise to

$$F = \frac{i\lambda}{V_{\Omega}} \left(\partial_{\mu}\varphi^* \int_{\Omega} \partial^{\mu}\varphi dV - \partial_{\mu}\varphi \int_{\Omega} \partial^{\mu}\varphi^* dV \right). \quad (66)$$

As shown in (66), F has an anti-symmetry with respect to $\partial^{\mu}\varphi$ and $\partial_{\mu}\varphi^*$. So we can easily verify that it satisfies the zero mean condition. That is

$$\int_{\Omega} F(x^{\mu}) dV = 0. \quad (67)$$

With (45), the nonlocal residual is also represented as

$$F = ie \left(1 + \frac{\delta U}{\delta\theta} \right) [(\partial^{\mu}\varphi)A_{\mu}\varphi^* - (\partial^{\mu}\varphi^*)A_{\mu}\varphi], \quad (68)$$

where $\theta = \partial_{\mu}\varphi\partial^{\mu}\varphi^*$, being an intermediate variable. Inserting (68) in (66) leads to

$$e \left(1 + \frac{\delta U}{\delta\theta} \right) [(\partial^{\mu}\varphi)A_{\mu}\varphi^* - (\partial^{\mu}\varphi^*)A_{\mu}\varphi] = \frac{\lambda}{V_{\Omega}} \left(\partial_{\mu}\varphi^* \int_{\Omega} \partial^{\mu}\varphi dV - \partial_{\mu}\varphi \int_{\Omega} \partial^{\mu}\varphi^* dV \right), \quad (69)$$

from which we immediately obtain

$$A_{\mu}\varphi = -\frac{\lambda}{eV_{\Omega}} \left(1 + \frac{\delta U}{\delta\theta} \right)^{-1} \int_{\Omega} \partial_{\mu}\varphi dV, \quad A_{\mu}\varphi^* = -\frac{\lambda}{eV_{\Omega}} \left(1 + \frac{\delta U}{\delta\theta} \right)^{-1} \int_{\Omega} \partial_{\mu}\varphi^* dV. \quad (70)$$

Equation (70) characterizes correlations between the gauge field A_{μ} and the gradient of the Lagrangian field $\partial_{\mu}\varphi$ under the case of global gauge invariance concerned with nonlocal effects. Obviously, if $U = U(\partial_{\mu}\varphi\partial^{\mu}\varphi^*) = 0$, then we easily verify $J_{,\mu}^{\mu} = 0$ from (48), (54) and (60). This shows that no nonlocal effect will exist, provided no fluctuation of self-energy occurs. So far, we have seen that although the fluctuation of self-energy of a field has no influence on the motion equation and the symmetry of the action integral defined on the global space-time domain, it enables to locally break the conservation flux so that it could not remain constant. This is an observable effect. We expect that the experiment in the future can confirm existence of this effect.

5 Conclusions

Under the condition that the localized assumption is no longer valid, the connection between the conservation flux and the nonlocal residual is established through the symmetry breaking of an action integral under the global gauge transformation. This is the so-called nonlocal balance equation. It shows that divergence of the conservation flux is equal to the nonlocal residual rather than zero. Due to this fact, the nonlocal balance equation is not a conservation law in a strict sense, but it should be regarded as a generalization of the Noether's theorem under the global gauge transformation.

The nonlocal residual is subjected to the constraint of the zero mean condition. Therefore, it has no influences on the motion equation and the gauge invariance of the action integral

defined on a bounded space-time domain Ω as a whole. However, the nonlocal residual enables to locally break the conservation flux so that it no longer remains constant in the local sub-domain of Ω . This is an observable symmetry breaking. In our opinion, some special experiments should be able to certify it.

Physically, the nonlocal residual may be interpreted as a new source originating from interactions within a field. On the basis of this argument, the correlation between the nonlocal residual and the gauge field is naturally established, just as shown by (45). This result also shows that if the localized assumption fails, then if and only if a gauge field is introduced, the global gauge symmetry and its breaking of a Lagrangian field can be accurately characterized.

The example given in this paper further verifies that the localized assumption is no avail under some circumstance. If such situations occur, the global gauge invariance of the action integral of a Lagrange field can be described by the nonlocal balance equation, by which the nonlocal residual and the gauge field are also determined explicitly. Meanwhile, this example also illustrates that the localized assumption probably fails only when self-interactions of field occur. For a free field, the localized assumption is always valid.

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